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AN INITIAL POST-BUCKLING ANALYSIS FOR PRISMATIC PLATE ASSEMBLIES UNDER AXIAL COMPRESSION

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Abstract—This paper provides a Koiter-type initial post-buckling analysis for prismatic plate assemblies made of isotropic materials. The structures are assumed to consist of a series of long flat strips rigidly connected together at their edges, subjected to longitudinal in-plane compressive stress. The transcendental eigenvalue problems, which arise when exact solutions to the member equations are used to form the stiffness matrix of the plate assemblies, are first solved to obtain the buckling load and corresponding buckling mode of the structure. The analysis then obtains exact solutions to the post-buckling member equations and the *a*-coefficient and *b*-coefficient which characterize the initial post-buckling behavior. The post-buckling characteristics of the stiffened plate are found to be influenced significantly by the height of the stiffener. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The buckling of prismatic structures consisting of a series of long, thin, flat isotropic plates rigidly connected together along their longitudinal edges was analyzed by Wittrick (1968). In a series of subsequent papers (Williams and Wittrick, 1969; Wittrick and Williams, 1970; Williams and Wittrick, 1972; Williams, 1972; Wittrick and Williams, 1974), Wittrick and Williams developed a theory and algorithms which can very efficiently solve buckling and vibration problems for such structures, both for isotropic and anisotropic materials, as follows.

Typical examples of such structures are shown in cross section in Fig. 1. These structures are composed of a series of long flat plates connected along their longitudinal

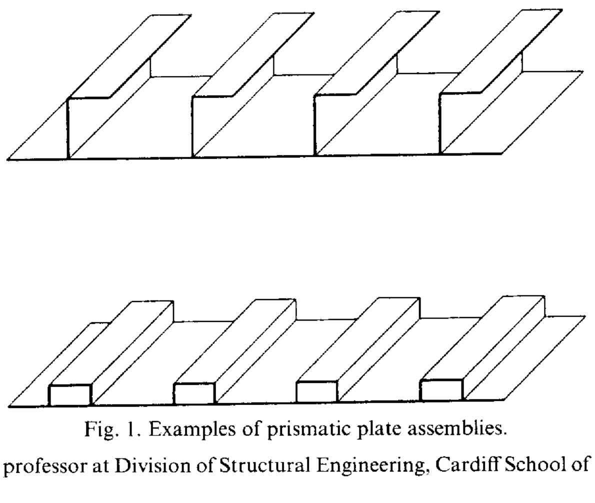

Fig. 1. Examples of prismatic plate assemblies.

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edges, so that each flat part of a stiffener, and each portion of skin between stiffeners, was represented by Williams and Wittrick as a single flat plate. As shown in Fig. 2, each plate was assumed to be subjected to a uniform longitudinal compression and it was also assumed that the mode of buckling varies sinusoidally in the longitudinal direction. During buckling the longitudinal edges of any component plate, where it is connected to neighboring plates, are subjected to perturbation forces and moments which are sinusoidally distributed along the edges, and these give rise to sinusoidally varying edge displacements and rotations. Wittrick and Williams (1974) demonstrated that explicit analytical expressions for the elements of an 8×8 stiffness matrix can be derived which convert the amplitudes of the edge displacements and rotations into the amplitudes of the edge forces and moments of the component plate element. By assembling such element stiffness matrices, using the usual procedures of the finite element method, including a transformation of axes for the plates representing stiffener webs, the eigenvalue problem finally can be obtained in the form

$$\mathbf{KD} = \mathbf{0}$$

where **K** is an overall stiffness matrix and **D** is a displacement vector containing the amplitudes of the longitudinally sinusoidally varying displacements of the four independent degrees of freedom of each of the longitudinal junctions between plates, as well as of the outer free or restrained (i.e. supported) edges. The elements of **K** are complicated transcendental functions of the thickness and width of each component plate, of its material properties, of the load factor, and also of the longitudinal half-wavelength λ of the sinusoidal variations. The algorithms of Wittrick and Williams enable any eigenvalue to be converged on systematically with complete certainty. The resulting method has been proven to be two–three orders of magnitude quicker than the conventional finite element method, as demonstrated by the optimum design software VICONOPT (Butler and Williams, 1992).

Structures consisting of prismatic plate assemblies, e.g. the stiffened plates shown in Fig. 1, are widely used in aircraft and ships. The post-buckling behavior of such structures can be influenced significantly by geometric parameters of the structure, e.g. the shape and spacing of stiffeners. Koiter (1945) showed that the initial post-buckling behavior of the idealized perfect structure determines the kind of buckling to be expected for the corresponding imperfect structure. Some structures exhibit stable post-buckling behavior and hence are imperfection insensitive, i.e. they can carry extra load after buckling occurs, while other are imperfection sensitive, i.e. small imperfections can make the snap-buckling load substantially lower than the critical buckling load of the idealized perfect structure. In current computer software for the optimum design of structures the structural mass is often taken as the objective function and must be minimized subject to buckling and other constraints (Butler and Williams, 1992). The present work is motivated by the fact that the post-buckling performance of the structure should ideally be considered, as well as these buckling constraints.

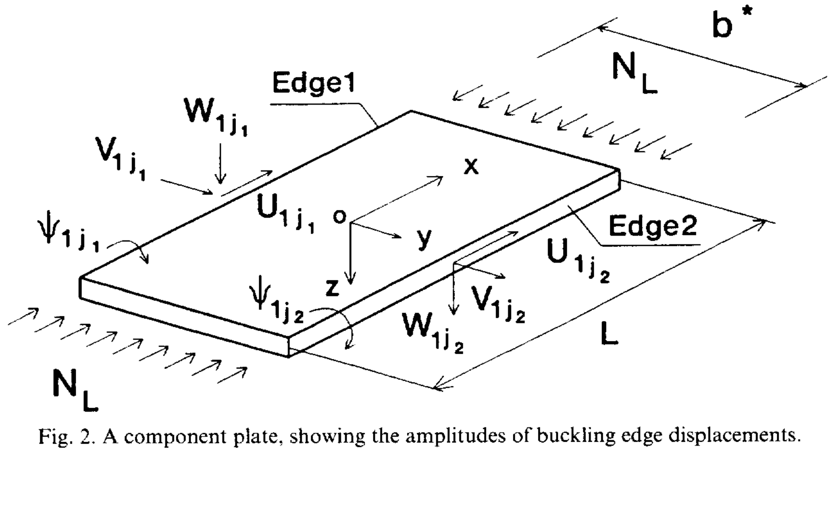

Fig. 2. A component plate, showing the amplitudes of buckling edge displacements.

During the late 1960s and early 1970s Koiter's general theory of elastic stability (Koiter, 1945), which was originally written in Dutch and was then translated into English in 1967, spawned much research into the initial post-buckling of thin-walled structures. An equivalent variation of Koiter's approach, using a version of the Principle of Virtual Work, was developed by Budiansky and Hutchinson (1964). Post-buckling behavior of stiffened cylindrical shells was analyzed by Hutchinson and Amazigo (1967). A unified, general presentation of Koiter's basic theory, employing the succinct notation of functional analysis, was provided by Budiansky (1974) who used both the energy and virtual work approaches. More recently, applications of Koiter's theory to the analysis of the post-buckling of composite laminated plates and shells have been reported by, e.g. Hui (1986), Sun and Hansen (1988), and Sun and Mao (1993). In such analyses, however, stiffeners are often "smeared-out", for both isotropic and anisotropic thin-walled structures.

Because the theory used by Wittrick and Williams makes the assumption that all three components of displacement vary sinusoidally along any longitudinal line, it is correct for infinitely long structures. For finite length structures it is implied either that the end cross sections are appropriately supported, so that exact results are again obtained, or that the half-wavelength of the mode is much smaller than the overall length of the structure, giving close approximations to the exact answer. These are the only assumptions made other than those inherent in thin plate theory. Hence, their method is referred to as "exact" member theory in the literature, because it solves the differential equations exactly and so requires only one element per plate, unlike the more usual approximate finite element methods which require many elements to obtain high accuracy.

In the present study, Koiter's theory is employed to analyze initial post-buckling behavior of prismatic plate assemblies which are composed of isotropic materials and are subjected only to longitudinal compressive load. The post-buckling equations are solved exactly, and the post-buckling coefficients are obtained by exact integration for all component plates. In this sense, the post-buckling analysis is also "exact", so that the results should be more accurate than those obtained by the commonly used "smeared-out" stiffener assumption. Example calculations are conducted for asymmetrically and symmetrically stiffened isotropic plates and these confirm that the post-buckling behavior depends significantly on the geometry parameter of the stiffeners considered.

2. BASIC FORMULATION

In order to deal adequately with all possible modes of buckling, the destabilizing effect of the basic longitudinal compressive load N_L on deformations in the plane of the isotropic plate must be properly accounted for. As in Wittrick (1968), the following pair of non-linear equations, which were originally developed by Novozhilov, are used for the in-plane equilibrium:

$$\left.\begin{aligned}\frac{\partial}{\partial x}\left[\left(1+\frac{\partial u}{\partial x}\right)N_x+\frac{\partial u}{\partial y}N_{xy}\right]+\frac{\partial}{\partial y}\left[\left(1+\frac{\partial u}{\partial x}\right)N_{yx}+\frac{\partial u}{\partial y}N_y\right]=0\\ \frac{\partial}{\partial y}\left[\left(1+\frac{\partial v}{\partial y}\right)N_y+\frac{\partial v}{\partial x}N_{yx}\right]+\frac{\partial}{\partial x}\left[\left(1+\frac{\partial v}{\partial y}\right)N_{xy}+\frac{\partial v}{\partial x}N_x\right]=0\end{aligned}\right\} \tag{1}$$

where the in-plane stress resultants N_x, N_y and $N_{xy}(=N_{yx})$ are for longitudinal tension, transverse tension and shear, respectively, and u and v are the mid-surface displacements in the x and y directions, respectively. The following non-linear partial differential equation is used for the out-of-plane equilibrium:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+N_y\frac{\partial^2 w}{\partial y^2} \tag{2}$$

where w is the out-of-plane deflection of the plate and the flexural rigidity of the plate is given by

$$D = \frac{Et^3}{12(1-v^2)}$$

where E is Young's modulus, v is Poisson's ratio and t is the thickness of the plate. The in-plane constitutive relationship for an isotropic plate is

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = A \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{3}$$

where the in-plane stiffness

$$A = Et/(1-v^2).$$

The von Kármán type non-linear strain–displacement relations used are

$$\left.\begin{aligned} \varepsilon_x &= u_{,x} + \tfrac{1}{2}(w_{,x})^2 \\ \varepsilon_y &= v_{,y} + \tfrac{1}{2}(w_{,y})^2 \\ \gamma_{xy} &= u_{,y} + v_{,x} + w_{,x}w_{,y} \end{aligned}\right\} \tag{4}$$

where the comma denotes partial differentiation, i.e. $(\ )_{,x} = \partial(\ )/\partial x$, etc.

Following the method outlined in Budiansky (1974), asymptotic perturbation expansions of the solution valid in the neighborhood of the bifurcation point are assumed in the form

$$\left.\begin{aligned} u &= u_0 + \delta u_1 + \delta^2 u_2 \\ v &= v_0 + \delta v_1 + \delta^2 v_2 \\ w &= w_0 + \delta w_1 + \delta^2 w_2 \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} N_x &= N_{x0} + \delta N_{x1} + \delta^2 N_{x2} \\ N_y &= N_{y0} + \delta N_{y1} + \delta^2 N_{y2} \\ N_{xy} &= N_{xy0} + \delta N_{xy1} + \delta^2 N_{xy2} \end{aligned}\right\}. \tag{6}$$

The subscripts 0, 1 and 2 indicate that the corresponding functions represent pre-buckling, buckling and post-buckling quantities, respectively. δ is the perturbation parameter defined by $\delta = w_{max}/t$, where w_{max} is the maximum out-of-plane displacement of the buckling mode.

In the pre-buckling stage for pure longitudinal compression,

$$\left.\begin{aligned} &N_{x0} = -N_L; \quad N_{y0} = 0; \quad N_{xy0} = 0; \\ &u_{0,y} = v_{0,x} = 0; \\ &u_{0,x} = -N_L/(Et); \quad v_{0,y} = vN_L/(Et) \end{aligned}\right\}. \tag{7}$$

It is assumed that there is no out-of-plane prebuckling displacement, i.e.

$$w_0 = 0. \tag{8}$$

Equations (7) and (8) give linear relationships between the fundamental solutions for stresses, strains and displacements, i.e. there is a linear pre-buckling state. As a perturbation procedure, substituting eqns (4)–(6) into eqn (3) and collecting the first order terms of δ gives

$$\left.\begin{aligned} N_{x1} &= A(u_{1,x}+\nu v_{1,y}) \\ N_{y1} &= A(\nu u_{1,x}+v_{1,y}) \\ N_{xy1} &= \frac{1-\nu}{2}A(u_{1,y}+v_{1,x}) \end{aligned}\right\}. \tag{9}$$

Also, by collecting the coefficients of δ^2,

$$\left.\begin{aligned} N_{x2} &= A[u_{2,x}+\tfrac{1}{2}(w_{1,x})^2]+\nu A[v_{2,y}+\tfrac{1}{2}(w_{1,y})^2] \\ N_{y2} &= \nu A[u_{2,x}+\tfrac{1}{2}(w_{1,x})^2]+A[v_{2,y}+\tfrac{1}{2}(w_{1,y})^2] \\ N_{xy2} &= \frac{1-\nu}{2}A(u_{2,y}+v_{2,x}+w_{1,x}w_{1,y}) \end{aligned}\right\}. \tag{10}$$

Substituting eqns (5) and (6) into eqn (1), taking eqn (7) into account and collecting the first order terms of δ leads to

$$\frac{\partial}{\partial x}\left[\left(1+\frac{\partial u_0}{\partial x}\right)N_{x1}+\frac{\partial u_1}{\partial x}N_{x0}\right]+\frac{\partial}{\partial y}\left[\left(1+\frac{\partial u_0}{\partial x}\right)N_{xy1}\right]=0$$

$$\frac{\partial}{\partial y}\left[\left(1+\frac{\partial v_0}{\partial y}\right)N_{y1}\right]+\frac{\partial}{\partial x}\left[\left(1+\frac{\partial v_0}{\partial y}\right)N_{xy1}+\frac{\partial v_1}{\partial x}N_{x0}\right]=0.$$

The prebuckling strains can be neglected because they are small compared to unity, i.e. $u_{0,x} \ll 1$ and $v_{0,y} \ll 1$. Hence, using eqn (7),

$$\left.\begin{aligned} \frac{\partial N_{x1}}{\partial x}+\frac{\partial N_{xy1}}{\partial y}-N_{\mathrm{L}}\frac{\partial^2 u_1}{\partial x^2} &= 0 \\ \frac{\partial N_{y1}}{\partial y}+\frac{\partial N_{xy1}}{\partial x}-N_{\mathrm{L}}\frac{\partial^2 v_1}{\partial x^2} &= 0 \end{aligned}\right\}. \tag{11}$$

The out-of-plane buckling equation is obtained by substituting eqns (5) and (6) into eqn (2) and collecting the coefficients of δ, to give

$$D\left(\frac{\partial^4 w_1}{\partial x^4}+2\frac{\partial^4 w_1}{\partial x^2\,\partial y^2}+\frac{\partial^4 w_1}{\partial y^4}\right)+N_{\mathrm{L}}\frac{\partial^2 w_1}{\partial x^2}=0. \tag{12}$$

Equations (11) and (12) constitute a set of linear buckling equations for the problem which are exactly the same as those in Wittrick (1968).

Substituting eqns (5) and (6) into eqn (1) and collecting the terms in δ^2 leads to the post-buckling in-plane equations

$$\frac{\partial}{\partial x}\left[\left(1+\frac{\partial u_0}{\partial x}\right)N_{x2}+\frac{\partial u_2}{\partial x}N_{x0}+\frac{\partial u_1}{\partial x}N_{x1}+\frac{\partial u_1}{\partial y}N_{xy1}\right]$$

$$+\frac{\partial}{\partial y}\left[\left(1+\frac{\partial u_0}{\partial x}\right)N_{xy2}+\frac{\partial u_1}{\partial x}N_{xy1}+\frac{\partial u_1}{\partial y}N_{y1}\right]=0$$

$$\frac{\partial}{\partial y}\left[\left(1+\frac{\partial v_0}{\partial y}\right)N_{y2}+\frac{\partial v_1}{\partial y}N_{y1}+\frac{\partial v_1}{\partial x}N_{xy1}\right]$$
$$+\frac{\partial}{\partial x}\left[\left(1+\frac{\partial v_0}{\partial y}\right)N_{xy2}+\frac{\partial v_2}{\partial x}N_{x0}+\frac{\partial v_1}{\partial y}N_{xy1}+\frac{\partial v_1}{\partial x}N_{x1}\right]=0.$$

As in the case of the in-plane buckling equations, $u_{0,x}$ and $v_{0,y}$ are negligible compared with unity. Similarly, further simplification can be made by neglecting the last two non-linear terms in each square bracket, so that the in-plane post-buckling equations become

$$\left.\begin{aligned}\frac{\partial N_{x2}}{\partial x}+\frac{\partial N_{xy2}}{\partial y}-N_{\mathrm{L}}\frac{\partial^2 u_2}{\partial x^2}=0\\ \frac{\partial N_{y2}}{\partial y}+\frac{\partial N_{xy2}}{\partial x}-N_{\mathrm{L}}\frac{\partial^2 v_2}{\partial x^2}=0\end{aligned}\right\}. \tag{13}$$

The out-of-plane post-buckling equation is obtained by substituting eqns (5) and (6) into eqn (2) and collecting the coefficients of δ^2, which gives

$$D\left(\frac{\partial^4 w_2}{\partial x^4}+2\frac{\partial^4 w_2}{\partial x^2\,\partial y^2}+\frac{\partial^4 w_2}{\partial y^4}\right)+N_{\mathrm{L}}\frac{\partial^2 w_2}{\partial x^2}=N_{x1}\frac{\partial^2 w_1}{\partial x^2}+2N_{xy1}\frac{\partial^2 w_1}{\partial x\,\partial y}+N_{y1}\frac{\partial^2 w_1}{\partial y^2}. \tag{14}$$

The basic two-dimensional buckling and post-buckling equations have now all been obtained.

3. BUCKLING PROBLEM

Equations (11) and (12) constitute a linear eigenvalue problem. Using the separation of variables method, they can be reduced to a set of one dimensional ordinary differential equations. Because the in-plane buckling displacements have been assumed to be sinusoidal along the longitudinal coordinate x with half-wave length λ, the separable forms of u_1 and v_1 are assumed to be

$$u_1(x,y)=U_1(Y)\sin X;\quad v_1(x,y)=V_1(Y)\cos X \tag{15}$$

where the non-dimensional variables X, Y and ω are defined as

$$X=\pi x/\lambda;\quad Y=\pi y/\lambda;\quad \omega=\pi b^*/\lambda. \tag{16}$$

The subscript 1 occurs because they are buckling mode variables and b^* is the width of the plate element, see Fig. 2. Substituting eqn (15) into eqns (9) and (11) leads to the pair of homogeneous ordinary differential equations

$$\left.\begin{aligned}(1-\nu)U_1''-2\left(1-\frac{N_{\mathrm{L}}}{A}\right)U_1-(1+\nu)V_1'=0\\ 2V_1''-\left(1-\nu-2\frac{N_{\mathrm{L}}}{A}\right)V_1+(1+\nu)U_1'=0\end{aligned}\right\}. \tag{17}$$

The superscript (′) denotes differentiation with respect to Y, e.g. $(\ )'=\mathrm{d}(\ )/\mathrm{d}Y$.

Solutions to eqns (17) are assumed in the form

$$\left.\begin{aligned} U_1(Y) &= (A_1 \cosh\theta Y - \phi B_1 \cosh\phi Y) + (A_2 \sinh\theta Y - \phi B_2 \sinh\phi Y) \\ V_1(Y) &= (-\theta A_1 \sinh\theta Y + B_1 \sinh\phi Y) + (-\theta A_2 \cosh\theta Y + B_2 \cosh\phi Y) \end{aligned}\right\} \tag{18}$$

where

$$\theta^2 = 1 - \frac{N_L}{A}; \quad \phi^2 = 1 - \frac{2(1+v)N_L}{(1-v^2)A} = 1 - 2(1+v)\frac{N_L}{Et} \tag{19}$$

and the normal displacement w is assumed to have the form

$$w_1(x,y) = W_1(Y)\cos X. \tag{20}$$

Substitution of eqn (20) into eqn (12) leads to the following fourth order ordinary differential equation for the normal buckling displacement function $W_1(Y)$:

$$W_1'''' - 2W_1'' + (1-\xi)W_1 = 0 \tag{21}$$

where

$$\xi = \frac{N_L \lambda^2}{\pi^2 D}. \tag{22}$$

For $\xi > 1$, the solution of eqn (21) can be written as

$$W_1(Y) = A_1^* \sinh q_1 Y + B_1^* \cosh q_1 Y + A_2^* \sin q_2 Y + B_2^* \cos q_2 Y \tag{23}$$

where

$$q_1 = \sqrt{1+\sqrt{\xi}}; \quad q_2 = \sqrt{\sqrt{\xi}-1}. \tag{24}$$

Alternatively, if $1 > \xi > 0$ the solution can be written as

$$W_1(Y) = A_1^* \sinh q_1 Y + B_1^* \cosh q_1 Y + A_2^* \sinh q_2 Y + B_2^* \cosh q_2 Y \tag{25}$$

where

$$q_1 = \sqrt{1+\sqrt{\xi}}; \quad q_2 = \sqrt{1-\sqrt{\xi}} \tag{26}$$

while for $\xi = 1$, the solution can be written as

$$W_1(Y) = A_1^* \sinh\sqrt{2}Y + B_1^* \cosh\sqrt{2}Y + A_2^* + B_2^* Y. \tag{27}$$

However, in practical situations, e.g. stiffened plates and plate assemblies with polygonal cross sections, only the case $\xi > 1$ has ever been encountered by the authors. Therefore, the associated computer program has no coding for the cases $\xi = 1$ and $1 > \xi > 0$ and would simply print a message if such cases were ever detected.

Following Wittrick and Williams (1974), the four unknown displacement amplitudes shown in Fig. 2 are assumed for each nodal line, so that

$$\mathbf{d}_j = \{\psi_{1j}, W_{1j}, V_{1j}, U_{1j}\} \quad (j = j_1, j_2). \tag{28}$$

Expressions for the exact stiffness matrix for this isotropic plate element are derived in Williams and Wittrick (1969). By applying these expressions to obtain the stiffness matrices of individual plates, the exact overall stiffness matrix **K** for the structure can be assembled by using the conventional routines of finite element analysis. The corresponding buckling problem can finally be expressed as the eigenvalue problem

$$\mathbf{KD} = \mathbf{0} \tag{29}$$

where the components of **K** are complicated transcendental functions of the longitudinal load N_L, and the vector **D** consists of the four displacement amplitudes of Fig. 2 for each nodal line.

The eigenvalue N_L can be obtained by the procedure given in Wittrick and Williams (1974). The corresponding eigenvector **D** is then obtained by the random force method described as the P_{RT} method in Hopper and Williams (1977). Next, the three displacement functions $W_1(Y)$, $V_1(Y)$, $U_1(Y)$ of each plate can be fully determined by substituting eqns (18) and (23) into the conditions

$$\left.\begin{aligned} U_1\left(-\frac{\omega}{2}\right) &= U_{1j_1}; \quad & U_1\left(\frac{\omega}{2}\right) &= U_{1j_2} \\ V_1\left(-\frac{\omega}{2}\right) &= V_{1j_1}; \quad & V_1\left(\frac{\omega}{2}\right) &= V_{1j_2} \\ W_1\left(-\frac{\omega}{2}\right) &= W_{1j_1}; \quad & W_1\left(\frac{\omega}{2}\right) &= W_{1j_2} \\ W_1'\left(-\frac{\omega}{2}\right) &= \frac{\lambda\psi_{1j_1}}{\pi}; \quad & W_1'\left(\frac{\omega}{2}\right) &= \frac{\lambda\psi_{1j_2}}{\pi} \end{aligned}\right\} \tag{30}$$

to give the four unknown constants A_1, B_1, A_2 and B_2 of eqn (18), and the four unknown constants A_1^*, B_1^*, A_2^* and B_2^* of eqn (23). The buckling load N_{Lc} and the buckling mode functions $U_1(Y)$, $V_1(Y)$ and $W_1(Y)$ for each component plate which are needed by the subsequent analysis, are now known, where N_{Lc} denotes the critical eigenvalue.

4. POST-BUCKLING PROBLEM

In-plane post-buckling displacements are assumed in the form

$$\left.\begin{aligned} u_2(x,y) &= u_0 x + U_2(Y)\sin 2X \\ v_2(x,y) &= v_0 y + V_{20}(Y) + V_2(Y)\cos 2X \end{aligned}\right\} \tag{31}$$

where the first term u_0x of $u_2(x,y)$ and v_0y of $v_2(x,y)$, exists because the second order field admits terms of the same form as the pre-buckling state. Substituting eqns (31) and (10) into eqn (13) gives a pair of second order ordinary differential equations for $U_2(Y)$ and $V_2(y)$

$$\begin{aligned} -4\left(1-\frac{N_{Lc}}{A}\right)U_2 + \frac{1-v}{2}U_2'' - (1+v)V_2' &= \frac{\pi}{2\lambda}\left[-W_1^2 + \frac{1+v}{2}W_1'^2 + \frac{1-v}{2}W_1W_1''\right] \\ V_2'' - 4\left(\frac{1-v}{2} - \frac{N_{Lc}}{A}\right)V_2 + (1+v)U_2' &= \frac{\pi}{2\lambda}[W_1W_1' - W_1'W_1''] \end{aligned} \tag{32}$$

and one differential equation for $V_{20}(Y)$

$$V''_{20} = -\frac{\pi}{2\lambda}[\nu W_1 W'_1 + W'_1 W''_1]. \tag{33}$$

Solutions to eqns (32) and (33) can be expressed as

$$\left.\begin{aligned} &U_2(Y) = U_{2h}(Y) + U_{2p}(Y); \quad V_2(Y) = V_{2h}(Y) + V_{2p}(Y); \\ &V_{20}(Y) = V_{20h}(Y) + V_{20p}(Y) \end{aligned}\right\} \tag{34}$$

where the functions with subscript "h" are general solutions of the corresponding homogeneous equations and those with subscript "p" are particular integrals of eqns (32) and (33). The method used to solve these ordinary differential equations can be found in typical textbooks, e.g. Ross (1980). General solutions to eqns (32) are assumed in the form

$$\left.\begin{aligned} U_{2h}(Y) &= \bar{A}_1 \cosh 2\theta Y - \phi\bar{B}_1 \cosh 2\phi Y + \bar{A}_2 \sinh 2\theta Y - \phi\bar{B}_2 \sinh 2\phi Y \\ V_{2h}(Y) &= -\theta\bar{A}_1 \sinh 2\theta Y + \bar{B}_1 \sinh 2\phi Y - \theta\bar{A}_2 \cosh 2\theta Y + \bar{B}_2 \cosh 2\phi Y \end{aligned}\right\} \tag{35}$$

and the general solution to eqn (33) can be written as

$$V_{20h} = \tilde{A}_1 + \tilde{A}_2 Y. \tag{36}$$

Now the right hand sides of eqns (32) and (33) are known functions which can be expanded by multiplication of functions W_1, W'_1 and W''_1, to become

$$\begin{aligned} &[C_1 + C_2 \cosh 2q_1 Y + C_3 \sinh q_1 Y \sin q_2 Y + C_4 \cosh q_1 Y \cos q_2 Y \\ &\quad + C_5 \cos 2q_2 Y] + [C_6 \sinh 2q_1 Y + C_7 \sinh q_1 Y \cos q_2 Y + C_8 \cosh q_1 Y \sin q_2 Y + C_9 \sin 2q_2 Y] \end{aligned} \tag{37}$$

where $C_1 - C_9$ are known constants but are different for eqns (32) and (33). The particular integrals U_{2p}, V_{2p} and V_{20p} take the same form as expression (37), and can be found by using the method of undetermined coefficients, as explained in detail in the Appendix. The five terms in the first bracket are even functions of Y for each component plate, while the four terms in the second bracket are odd ones. This grouping simplifies the final integrations for the post-buckling coefficient because, since the origin of the local coordinate is located in the middle of the plate, the odd functions integrate to zero.

The post-buckling out-of-plane function is assumed to have the separable form

$$w_2(x, y) = W_{20}(Y) + W_2(Y) \cos 2X. \tag{38}$$

Substituting eqns (38), (9) and (15) into eqn (14) gives the following fourth order ordinary differential equations for $W_{20}(Y)$ and $W_2(Y)$

$$\begin{aligned} W''''_2 - 8W''_2 + 4\left(4 - \frac{\lambda^2 N_{Lc}}{\pi^2 D}\right) W_2 = \frac{6\lambda}{t^2\pi}[&-(U_1 + \nu V'_1)W_1 \\ &+ (1-\nu)(U'_1 - V_1)W'_1 + (\nu U_1 + V'_1)W''_1] \end{aligned} \tag{39}$$

$$W''''_{20} = \frac{6\lambda}{t^2\pi}[-(U_1 + \nu V'_1)W_1 - (1-\nu)(U'_1 - V_1)W'_1 + (\nu U_1 + V'_1)W''_1]. \tag{40}$$

Solutions to eqns (39) and (40) can be expressed as

$$W_2(Y) = W_{2h}(Y)+W_{2p}(Y); \quad W_{20}(Y) = W_{20h}(Y)+W_{20p}(Y) \tag{41}$$

where the functions with subscript "h" are general solutions of the corresponding homogeneous equations and those with subscript "p" are particular integrals of eqns (39) and (40).

For $\xi > 4$, the general solution of eqn (39) can be written as

$$W_{2h}(Y) = \bar{A}_1^* \sinh \bar{q}_1 Y + \bar{B}_1^* \cosh \bar{q}_1 Y + \bar{A}_2^* \sin \bar{q}_2 Y + \bar{B}_2^* \cos \bar{q}_2 Y \tag{42}$$

where

$$\bar{q}_1 = \sqrt{2(2+\sqrt{\xi})}; \quad \bar{q}_2 = \sqrt{2(\sqrt{\xi}-2)}. \tag{43}$$

Alternatively, if $4 > \xi > 0$ the solution can be written as

$$W_{2h}(Y) = \bar{A}_1^* \sinh \bar{q}_1 Y + \bar{B}_1^* \cosh \bar{q}_1 Y + \bar{A}_2^* \sinh \bar{q}_2 Y + \bar{B}_2^* \cosh \bar{q}_2 Y \tag{44}$$

where

$$\bar{q}_1 = \sqrt{2(2+\sqrt{\xi})}; \quad \bar{q}_2 = \sqrt{2(2-\sqrt{\xi})}. \tag{45}$$

The right hand sides of eqns (39) and (40) consist of known functions which can be expanded as

$$\begin{aligned}
&[C_1 \cosh(\theta+q_1)Y + C_2 \cosh(\theta-q_1)Y + C_3 \cosh(\phi+q_1)Y + C_4 \cosh(\phi-q_1)Y \\
&\quad + C_5 \sinh\theta Y \sin q_2 Y + C_6 \cosh\theta Y \cos q_2 Y + C_7 \sinh\phi Y \sin q_2 Y \\
&\quad + C_8 \cosh\phi Y \cos q_2 Y] + [C_9 \sinh(\theta+q_1)Y + C_{10} \sinh(\theta-q_1)Y \\
&\quad + C_{11} \sinh(\phi+q_1)Y + C_{12} \sinh(\phi-q_1)Y + C_{13} \sinh\theta Y \cos q_2 Y \\
&\quad + C_{14} \cosh\theta Y \sin q_2 Y + C_{15} \sinh\phi Y \cos q_2 Y + C_{16} \cosh\phi Y \sin q_2 Y]
\end{aligned} \tag{46}$$

where C_1–C_{16} are known constants, and θ and ϕ are given by eqn (19) with N_L replaced by N_{Lc}. The eight terms in the first bracket are even functions of Y, while those in the second bracket are odd ones. The particular integrals W_{2p} and W_{20p} take the same form as expression (46) and can be solved by using the method of undetermined coefficients, as explained in the Appendix.

The kinematic boundary conditions for the amplitudes of the sinusoidal components of post-buckling displacements, $U_2(Y)$, $V_2(Y)$ and $W_2(Y)$ are

$$\left.\begin{aligned}
U_2\left(-\frac{\omega}{2}\right) &= U_{2j_1}; \quad U_2\left(\frac{\omega}{2}\right) = U_{2j_2} \\
V_2\left(-\frac{\omega}{2}\right) &= V_{2j_1}; \quad V_2\left(\frac{\omega}{2}\right) = V_{2j_2} \\
W_2\left(-\frac{\omega}{2}\right) &= W_{2j_1}; \quad W_2\left(\frac{\omega}{2}\right) = W_{2j_2} \\
W_2'\left(-\frac{\omega}{2}\right) &= \frac{\lambda\psi_{2j_1}}{\pi}; \quad W_2'\left(\frac{\omega}{2}\right) = \frac{\lambda\psi_{2j_2}}{\pi}
\end{aligned}\right\} \tag{47}$$

where the quantities with subscript "j_1" and "j_2" are values of $U_2(Y)$, $V_2(Y)$, $W_2(Y)$ and

$\psi_2(Y)$ at nodal lines of the left and right edge of the flat plate element, respectively (Fig. 3). The corresponding edge forces are given by (Wittrick and Williams, 1974)

$$\left.\begin{aligned}
M_{2j_1} &= -D\left(\frac{\pi}{\lambda}\right)^2\left[W_2''\left(-\frac{\omega}{2}\right)-4\nu W_{2j_1}\right]\\
Q_{2j_1} &= D\left(\frac{\pi}{\lambda}\right)^3\left[W_2'''\left(-\frac{\omega}{2}\right)-4(2-\nu)\frac{\lambda}{\pi}\psi_{2j_1}\right]\\
N_{2j_1} &= -\frac{\pi A}{\lambda}\left[V_2'\left(-\frac{\omega}{2}\right)+2\nu U_{2j_1}\right]-\frac{A}{4}\left(\frac{\pi}{\lambda}\right)^2\left[-\nu W_1^2\left(-\frac{\omega}{2}\right)+W_1'^2\left(-\frac{\omega}{2}\right)\right]\\
T_{2j_1} &= -\frac{1-\nu}{2}\frac{\pi A}{\lambda}\left[U_2'\left(-\frac{\omega}{2}\right)-2V_{2j_1}\right]-\frac{1-\nu}{4}A\left(\frac{\pi}{\lambda}\right)^2\left[W_1\left(-\frac{\omega}{2}\right)W_1'\left(-\frac{\omega}{2}\right)\right]\\
M_{2j_2} &= D\left(\frac{\pi}{\lambda}\right)^2\left[W_2''\left(\frac{\omega}{2}\right)-4\nu W_{2j_2}\right]\\
Q_{2j_2} &= -D\left(\frac{\pi}{\lambda}\right)^3\left[W_2'''\left(\frac{\omega}{2}\right)-4(2-\nu)\frac{\lambda}{\pi}\psi_{2j_2}\right]\\
N_{2j_2} &= \frac{\pi A}{\lambda}\left[V_2'\left(\frac{\omega}{2}\right)+2\nu U_{2j_2}\right]+\frac{A}{4}\left(\frac{\pi}{\lambda}\right)^2\left[-\nu W_1^2\left(\frac{\omega}{2}\right)+W_1'^2\left(\frac{\omega}{2}\right)\right]\\
T_{2j_2} &= \frac{1-\nu}{2}\frac{\pi A}{\lambda}\left[U_2'\left(\frac{\omega}{2}\right)-2V_{2j_2}\right]+\frac{1-\nu}{4}A\left(\frac{\pi}{\lambda}\right)^2\left[W_1\left(\frac{\omega}{2}\right)W_1'\left(\frac{\omega}{2}\right)\right]
\end{aligned}\right\}. \tag{48}$$

Solutions of differential equations (32) and (39) satisfying the kinematic boundary condition (47) can be obtained analytically, thereby giving $U_2(Y)$, $V_2(Y)$ and $W_2(Y)$ in terms of the edge displacements

$$\bar{\mathbf{d}}_j = \{\psi_{2j}, W_{2j}, V_{2j}, U_{2j}\}^T \quad (j = j_1, j_2). \tag{49}$$

Equations (48) then give the edge forces

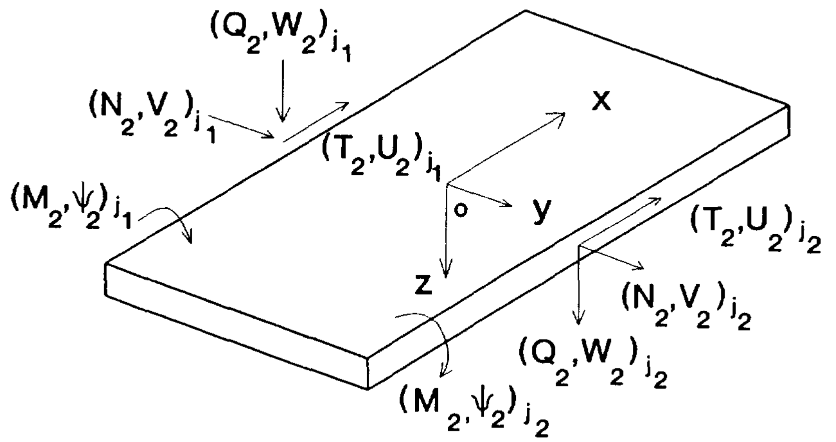

Fig. 3. A component plate, showing the amplitudes of sinusoidal post-buckling edge displacements and forces.

$$\bar{\mathbf{p}}_j = \{M_{2j}, Q_{2j}, N_{2j}, T_{2j}\}^T \quad (j = j_1, j_2) \tag{50}$$

in terms of the edge displacements and a set of linear simultaneous equations can be obtained for the element in the form

$$\bar{\mathbf{p}} = \overline{\mathbf{k}\mathbf{d}} + \bar{\mathbf{f}} \tag{51}$$

where

$$\bar{\mathbf{p}} = \{\bar{\mathbf{p}}_{j_1}, \bar{\mathbf{p}}_{j_2}\}^T; \quad \bar{\mathbf{d}} = \{\bar{\mathbf{d}}_{j_1}, \bar{\mathbf{d}}_{j_2}\}^T, \quad \bar{\mathbf{f}} = \{\bar{\mathbf{f}}_{j_1}, \bar{\mathbf{f}}_{j_2}\}^T \tag{52}$$

and $\bar{\mathbf{f}}$ consists of terms involving values of particular integrals at nodal lines. Expressions for the stiffness matrix $\bar{\mathbf{k}}$ of the post-buckling problem are in the same form as those of the buckling problem given in Williams and Wittrick (1969), except λ is replaced by $\lambda/2$. By applying these expressions to obtain the stiffness matrices of individual plates, the overall stiffness matrix $\bar{\mathbf{K}}$ for the structure can be assembled by using the conventional routines of finite element analysis. The corresponding problem can finally be expressed as

$$\overline{\mathbf{K}\mathbf{D}} = \bar{\mathbf{F}} \tag{53}$$

where $\bar{\mathbf{D}}$ is the displacement vector containing the four degrees of freedom of each of the nodal lines. Once the displacements at nodal lines are solved for, the general solutions $U_{2h}(Y)$, $V_{2h}(Y)$ and $W_{2h}(Y)$ can be analytically determined.

The non-sinusoidal components of post-buckling displacements $V_{20}(Y)$ of eqn (31) and $W_{20}(Y)$ of eqn (38) can be solved for by the same procedure. The stiffness matrix that results has the same form as that of beam and truss structures, and is not repeated in the present work.

Two constants u_0 and v_0 of eqn (31) can be determined using the method described by Hui (1983). They have the form

$$u_0 = \frac{\nu R_2 - R_1}{2S_b(1-\nu^2)}; \quad v_0 = \frac{\nu R_1 - R_2}{2S_b(1-\nu^2)} \tag{54}$$

where S_b is the summation of the widths of all plate elements,

$$\left.\begin{aligned} R_1 &= \sum_1^N \int_{-\omega/2}^{\omega/2} \left[\frac{\pi}{2\lambda}(W_1^2 + \nu W_1'^2) + 2\nu V_{20}'\right] \mathrm{d}Y \\ R_2 &= \sum_1^N \int_{-\omega/2}^{\omega/2} \left[\frac{\pi}{2\lambda}(\nu W_1^2 + W_1'^2) + 2V_{20}'\right] \mathrm{d}Y \end{aligned}\right\} \tag{55}$$

and N is the number of plate elements.

5. POST-BUCKLING COEFFICIENTS

Following the theory outlined in Hutchinson and Amazigo (1967), the load N_L in the vicinity of the buckling load N_{Lc} can be expressed as

$$\frac{N_L}{N_{Lc}} = 1 + a\delta + b\delta^2 + \cdots \tag{56}$$

where a and b are post-buckling coefficients. Prismatic plate assemblies may tend to buckle in a particular direction under longitudinal compression, i.e. the equilibrium path may not be independent of the sign of the amplitude of the buckling mode. Hence, the structure will fail when N_L is less than the buckling load N_{Lc} for one sign of the imperfection amplitude, but will have a reserve of strength above N_{Lc} when the imperfection has the opposite sign. Such a structure is classified as an unstable asymmetric system and the coefficient a does not vanish. Since δ is much less than unity in the vicinity of the buckling load, the post-buckling behavior can be depicted well by the first two terms of the right hand side of eqn (37), i.e. the δ^2 term can be neglected. For other structures, such as a flat isotropic plate or circular cylinder, the post-buckling path is independent of the sign of the amplitude of the buckling mode and is symmetric about the point of bifurcation. In this case, coefficient a vanishes and the initial post-buckling behavior of the structure hinges on the sign and amplitude of b. A negative b means that the load carrying ability diminishes immediately after buckling and the structure is imperfection sensitive, while a positive b means that the structure has the ability to withstand increasing load after buckling.

The expression for a is (see Budiansky, 1974)

$$a = -\frac{3}{2}\frac{a_1}{a_2} \tag{57}$$

where

$$\left.\begin{aligned} a_1 &= \iint \{N_{x1} w_{1,x}^2 + N_{y1} w_{1,y}^2 + 2N_{xy1} w_{1,x} w_{1,y}\}\, \mathrm{d}S \\ a_2 &= \iint N_{Lc} w_{1,x}^2\, \mathrm{d}S \end{aligned}\right\}. \tag{58}$$

The integration is over the mid-surface S of every plate element. It is assumed that axially there are m half waves, i.e. $m = L/\lambda$ and L is the length of the plate. Substitution of eqns (9), (15) and (20) into the integrands of eqn (58) gives

$$a = \begin{cases} 0 & m \text{ is even} \\ +a_0 & m = 1, 5, 9, \ldots \\ -a_0 & m = 3, 7, 11, \ldots \end{cases} \tag{59}$$

where

$$a_0 = \frac{2Et\pi}{\lambda(1-\nu^2)N_{Lc}} \times \frac{\sum_1^N \int_{-\omega/2}^{\omega/2} a_1(Y)\,\mathrm{d}Y}{\sum_1^N \int_{-\omega/2}^{\omega/2} W_1^2\,\mathrm{d}Y}$$

and

$$a_1(Y) = (U_1 + \nu V_1')W_1^2 + 2(\nu U_1 + V_1')W_1'^2 - (1-\nu)(U_1' - V_1)W_1 W_1'. \tag{60}$$

When $a = 0$, b is given by [see Budiansky (1974)]

$$b = -\frac{b_1}{b_2} \tag{61}$$

where

$$\begin{aligned} b_1 &= \iint \{2N_{x1} w_{1,x} w_{2,x} + 2N_{y1} w_{1,y} w_{2,y} + 2N_{xy1}(w_{1,x} w_{2,y} + w_{2,x} w_{1,y}) \\ &\quad + N_{x2} w_{1,x}^2 + N_{y2} w_{1,y}^2 + 2N_{xy2} w_{1,x} w_{1,y}\}\, \mathrm{d}S \\ b_2 &= a_2 (= \iint N_{Lc} w_{1,x}^2\, \mathrm{d}S). \end{aligned} \tag{62}$$

Substitution of eqns (9), (10), (15) and (20) into the integrands leads to

$$b = \frac{Et\pi}{2\lambda(1-\nu^2)N_{Lc}} \times \frac{\sum_1^N \int_{-\omega/2}^{\omega/2} b_1(Y)\,\mathrm{d}Y}{\sum_1^N \int_{-\omega/2}^{\omega/2} W_1^2\,\mathrm{d}Y} \tag{63}$$

where

$$\begin{aligned} b_1(Y) = & -4(U_1+\nu V_1')W_1W_2+2(\nu U_1+V_1')W_1'(W_2'+2W_{20}') \\ & +2(1-\nu)(U_1'-V_1)W_1(W_2'-2W_{20}')-2(1-\nu)(U_1'-V_1)W_1'W_2 \\ & -(2U_2+\nu V_2')W_1^2+2\nu V_{20}'W_1^2+(2\nu U_2+V_2')W_1'^2+2V_{20}'W_1'^2 \\ & -(1-\nu)(U_2'-2V_2)W_1W_1'+\frac{3\pi}{4\lambda}W_1'^4+\frac{3\pi}{4\lambda}W_1^4+\frac{\pi}{2\lambda}W_1^2W_1'^2. \end{aligned} \tag{64}$$

The expansion of the functions $a_1(Y)$ and $b_1(Y)$ of eqns (60) and (64) involves quite complicated algebraic manipulation, despite the advantage that only even functions need to be retained since the odd functions integrate to zero over the width of the plate, i.e. for the region $-\omega/2 \leqslant Y \leqslant \omega/2$.

6. EXAMPLES AND DISCUSSIONS

The theory and associated computer code were tested by application to a simply supported isotropic rectangular plate with length $\tilde{\mathbf{a}}$, width $\tilde{\mathbf{b}}$ and thickness t. The classical buckling load of the plate is

$$N_{Lc} = \left(\alpha+\frac{1}{\alpha}\right)^2 \frac{Et^3\pi^2}{12\tilde{b}^2(1-\nu^2)} \tag{65}$$

where $\alpha = \tilde{a}/\tilde{b}$. The post-buckling of this isotropic flat plate obviously belongs to the class of symmetric bifurcation and so the coefficient a vanishes. Based on the von Kármán equations, Budiansky (1974) provided the following formula for the post-buckling coefficient of a simply supported isotropic square plate ($m = 1$):

$$b = \tfrac{3}{8}(1-\nu^2). \tag{66}$$

However, it is easy to obtain the post-buckling coefficient b of a rectangular plate by slightly modifying Budiansky's formulation, as

$$b = \frac{3\left(\alpha^2+\frac{1}{\alpha^2}\right)(1-\nu^2)}{4\left(\alpha+\frac{1}{\alpha}\right)^2}. \tag{67}$$

In order to verify the proposed method, the plate was divided into two, four, 10 and 50 strip elements of equal width, and unequally divided into three strips with successive widths of 0.25 $\tilde{b}$, 0.25 $\tilde{b}$ and 0.5 $\tilde{b}$, giving five cases in all. Thus, a nodal line always coincided with the center line of the cross section of the plate, on which the maximum out-of-plane displacement w_{max} occurs. In the eigenvector, w_{max} was normalized to be equal to the thicknesses t of the plate. The critical buckling load given by each of the five cases agreed exactly with that given by eqn (65). The value of b was exactly the same for all five cases for a given t, but changed slightly with t and approached the value given by eqn (67) for

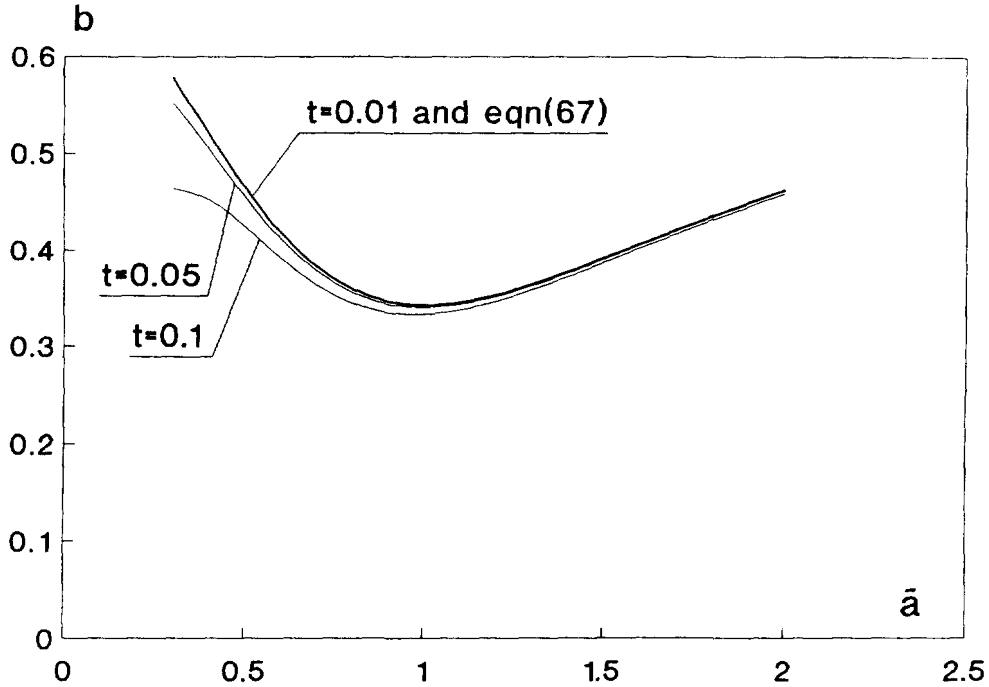

Fig. 4. b Coefficients for simply supported rectangular plates.

$t \to 0$. Figure 4 gives b coefficients for isotropic rectangular plates with width $\tilde{b} = 1$, thicknesses $t = 0.1$, 0.05 and 0.01, Poissons's ratio $\nu = 0.3$ and axial half-wave number $m = 1$, plotted against plate length $\bar{a}$. It is seen that as the thickness t decreases the value of b approaches that given by eqn (67) rapidly. The reason is that in the in-plane post-buckling eqn (32) the value of N_{Lc}/A vanishes as $t \to 0$, since substituting from eqn (65) and for the A of eqn (3) gives

$$\frac{N_{Lc}}{A} = \left(\alpha + \frac{1}{\alpha}\right)^2 \frac{t^2\pi^2}{12\tilde{b}^2}. \tag{68}$$

Then the terms in N_{Lc}/A in eqn (32) should be ignored and hence, the post-buckling analysis presented in the present work reduces to exactly that based on von Kármán's equations, and the postbuckling coefficient b reduces to exactly the value given by eqn (67). Thus, the method presented is justified.

The buckling and post-buckling behaviors of the two types of stiffened plate shown in Fig. 5 were also investigated. (Note that a node has been introduced half way between the stiffeners as the simplest way to get w_{max}.) The type I plate is eccentrically stiffened while the type II plate is symmetrically stiffened. The plates and their stiffeners all have thickness $t = 0.005\ B$. The total width of the plate is 4.0 B and its length is $L = 4.0\ B$. Poisson's ratio $\nu = 0.3$ and the height of the stiffeners is h. The simply supported boundary conditions on the two longitudinal edges are $V = W = 0$. In the longitudinal x direction, the half wave number m has to be prescribed, so that the axial half-wave length λ is determined. Then the critical buckling line load N_{Lc}, which is the lowest eigenvalue among those corresponding to several consecutive values of m, is obtained by solving the eigenproblem of eqns (17) and (21).

The dimensionless critical buckling line load N_{Lc}/EB for the type I plate is plotted against h/B in the top half of Fig. 6, on which m is the number of longitudinal half-waves, and the post-buckling coefficients are plotted in the bottom half of Fig. 6. For both halves of Fig. 6, the curves are composed of two segments, corresponding to the two different

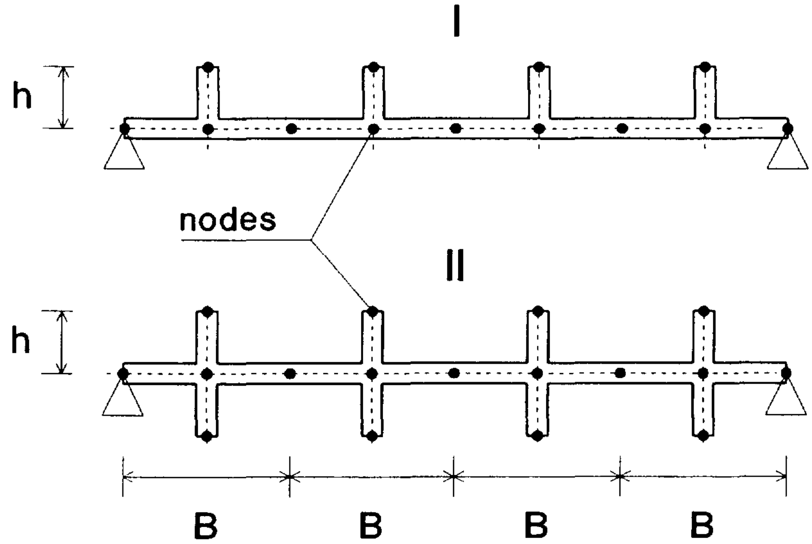

Fig. 5. Cross sections of two stiffened plates: (I) Type I and; (II) Type II.

buckling modes given by $m = 1$ and $m = 4$. The buckling stress increases rapidly as the stiffener height h increases for $0.05 \leqslant h/B < 0.085$ and the mode is the global one for which $m = 1$, which also has one "half-wave" in the y direction. The post-buckling coefficient a does not vanish since the buckling mode is an asymmetrical one when $m = 1$. When $0.085 < h/B \leqslant 0.5$, the critical buckling mode becomes a local one with $m = 4$ and with a equal to zero since the buckling mode is antisymmetric with respect to the mid-length of the structure. The coefficient b is plotted for this range of h/B because when $a = 0$, the post-buckling behavior is dominated by the coefficient b, see eqn (56). It should be noted from the upper half of Fig. 6 that, when $m = 4$, the buckling stress first increases slightly and then decreases with increase of the stiffener height h. This is consistent with the well-known fact that the stiffener, which is itself a thin plate, tends to buckle locally when it becomes wider and so triggers buckling of the structure.

Figure 7 shows the dimensionless critical buckling line load N_{Lc}/EB and post-buckling coefficient b for the symmetrically stiffened type II plate of Fig. 5. In this case the value of the coefficient a is always zero. Both the curves for the buckling load and for the b coefficient can be divided into three regions, corresponding to $m = 1$, 4 and 3. When $0.05 \leqslant h/B < 0.067$, the buckling line load increases rapidly as the stiffener height h is increased. In this region the plate buckles in a global mode with $m = 1$. When $0.067 < h/B \leqslant 0.5$, the buckling mode is local, with $m = 4$ and $m = 3$ successively. For all three modes, the value of b is positive, indicating stable post-buckling behavior of the symmetrically stiffened plate, so that a reserve of strength exists after the buckling load N_{Lc} is exceeded.

7. CONCLUSIONS

Based on Wittrick and Williams' theory for the eigenproblem, and Koiter's general theory of elastic stability, the present work provides an accurate analysis for the initial post-buckling of isotropic prismatic plate assemblies. Numerical examples for asymmetrically and symmetrically stiffened isotropic plates demonstrate that their post-buckling behavior is influenced significantly by the geometry of the stiffeners.

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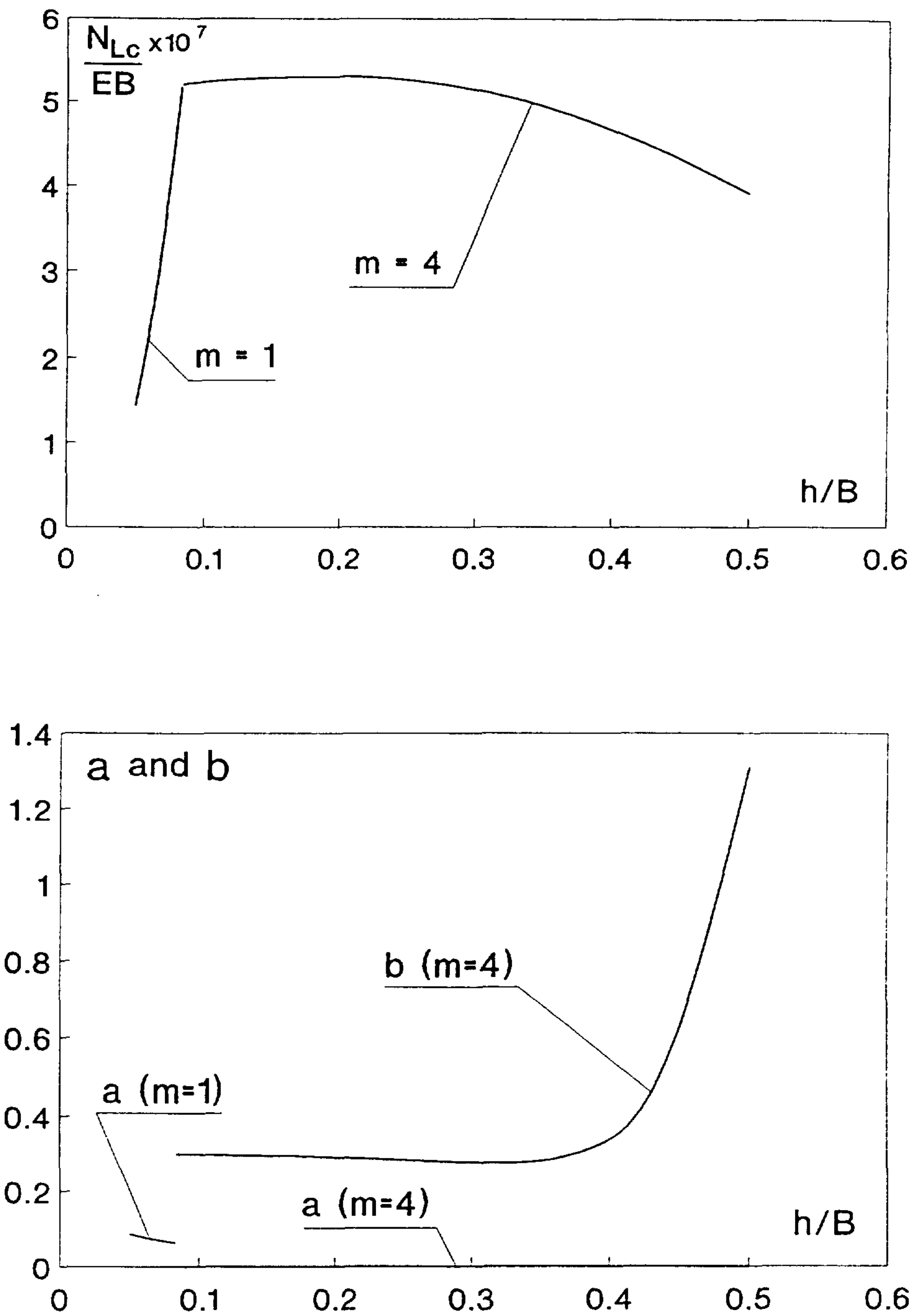
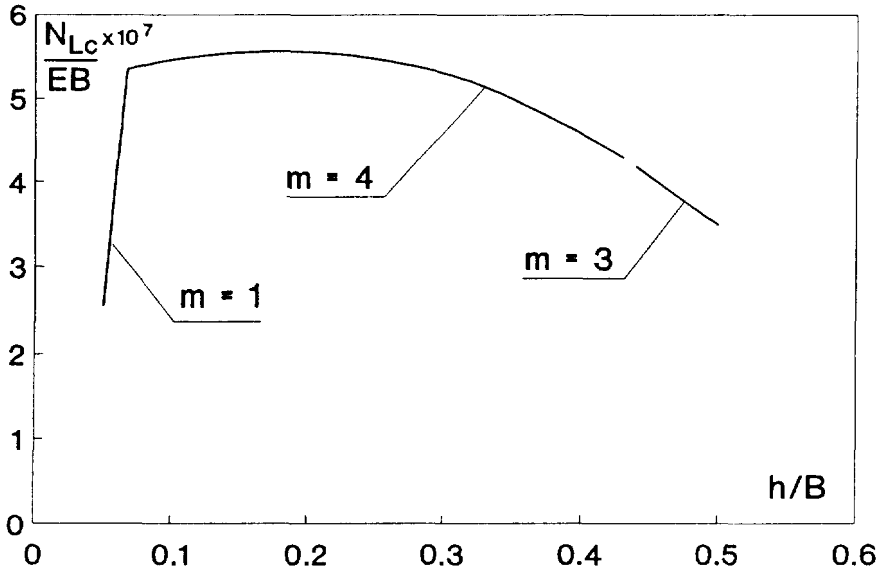


Fig. 6. Buckling load and post-buckling coefficients a and b of eqn (56) for Type I stiffened plate.



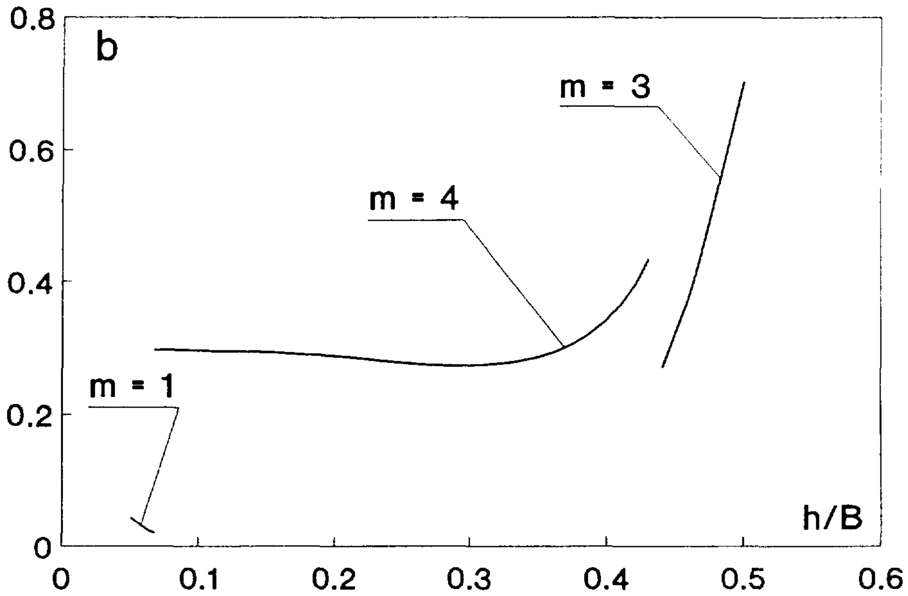


Fig. 7. Buckling load and b coefficient for Type II stiffened plate.

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APPENDIX

1. *Solution of the particular integrals of eqn* (32) *for* U_{2p} *and* V_{2p}

Let the constants a_1 and a_2 represent C_1 when eqn (37) is the right hand side of, respectively, the first and second of eqns (32). Clearly, the corresponding constants in the solution of U_{2p} and V_{2p} are

$$\frac{-a_1}{4\left(1-\dfrac{N_{Lc}}{A}\right)} \quad \text{and} \quad \frac{-a_2}{4\left(\dfrac{1-v}{2}-\dfrac{N_{Lc}}{A}\right)} \tag{A1}$$

respectively.

Let a_1 and a_2 instead represent C_2 and C_6 (or, C_6 and C_2) when eqn (37) is the right hand side of, respectively, the first and second of eqns (32). Then for each term $a_1 \cosh 2q_1 Y$ (or $a_1 \sinh 2q_1 Y$) in the first equation and $a_2 \sinh 2q_1 Y$ (or $a_2 \cosh 2q_1 Y$) in the second equation, there is a corresponding term $d_1 \cosh 2q_1 Y$ (or $d_1 \sinh 2q_1 Y$) in the solution $U_{2p}(Y)$ and a term $d_2 \sinh 2q_1 Y$ (or $d_2 \cosh 2q_1 Y$) in the solution $V_{2p}(Y)$, respectively, where

$$d_1 = \frac{a_1\gamma_1 + a_2\beta_1}{\beta_1^2 + \alpha_1\gamma_1}; \quad d_2 = \frac{a_2\alpha_1 - a_1\beta_1}{\beta_1^2 + \alpha_1\gamma_1} \tag{A2}$$

and

$$\left.\begin{aligned} \alpha_1 &= -4\left(1-\frac{N_{Lc}}{A}\right)+2(1-v)q_1^2 \\ \beta_1 &= 2(1+v)q_1 \\ \gamma_1 &= 4q_1^2 - 4\left(\frac{1-v}{2}-\frac{N_{Lc}}{A}\right) \end{aligned}\right\}. \tag{A3}$$

Let a_1 and a_2 instead represent C_9 and C_5 when eqn (37) is the right hand side of, respectively, the first and second of eqns (32). Then for each term $a_1 \sin 2q_2 Y$ in the first equation, and $a_2 \cos 2q_2 Y$ in the second equation, there is a corresponding term $d_1 \sin 2q_2 Y$ in the solution $U_{2p}(Y)$ and a term $d_2 \cos 2q_2 Y$ in the solution $V_{2p}(Y)$, respectively, where

$$d_1 = \frac{a_1\gamma_2 + a_2\beta_2}{\beta_2^2 - \alpha_2\gamma_2}; \quad d_2 = \frac{a_1\beta_2 + a_2\alpha_2}{\beta_2^2 - \alpha_2\gamma_2} \tag{A4}$$

and

$$\left.\begin{aligned} \alpha_2 &= 4\left(1-\frac{N_{Lc}}{A}\right)+2(1-v)q_2^2 \\ \beta_2 &= 2(1+v)q_2 \\ \gamma_2 &= 4q_2^2 + 4\left(\frac{1-v}{2}-\frac{N_{Lc}}{A}\right) \end{aligned}\right\}. \tag{A5}$$

Let a_1 and a_2 instead represent C_5 and C_9 when eqn (37) is the right hand side of, respectively, the first and second of eqns (32). Then for each term $a_1 \cos 2q_2 Y$ in the first equation, and $a_2 \sin 2q_2 Y$ in the second equation, there is a corresponding term $d_1 \cos 2q_2 Y$ in the solution $U_{2p}(Y)$ and a term $d_2 \sin 2q_2 Y$ in the solution $V_{2p}(Y)$, respectively, where

$$d_1 = \frac{a_1\gamma_2 - a_2\beta_2}{\beta_2^2 - \alpha_2\gamma_2}; \quad d_2 = \frac{-a_1\beta_2 + a_2\alpha_2}{\beta_2^2 - \alpha_2\gamma_2}. \tag{A6}$$

Let a_1 and a_2 instead represent C_3 and C_4, and a_3 and a_4 represent C_7 and C_8, when eqn (37) is the right hand side of, respectively, the first and second of eqns (32). Then for

$$a_1 \sinh q_1 Y \sin q_2 Y + a_2 \cosh q_1 Y \cos q_2 Y$$

at the right hand side of the first equation and for

$$a_3 \sinh q_1 Y \cos q_2 Y + a_4 \cosh q_1 Y \sin q_2 Y$$

at the right hand side of the second equation of eqns (32), there are corresponding terms

$$d_1 \sinh q_1 Y \sin q_2 Y + d_2 \cosh q_1 Y \cos q_2 Y$$

in the solution $U_{2p}(Y)$ and

$$d_3 \sinh q_1 Y \cos q_2 Y + d_4 \cosh q_1 Y \sin q_2 Y$$

in the solution of $V_{2p}(Y)$, respectively, where d_1–d_4 can be obtained by solving the algebraic equations

$$\left.\begin{aligned} \alpha_{11}d_1 - \alpha_{12}d_2 + \alpha_{13}d_3 - \alpha_{14}d_4 &= a_1 \\ \alpha_{12}d_1 + \alpha_{22}d_2 - \alpha_{14}d_3 - \alpha_{13}d_4 &= a_2 \\ \alpha_{13}d_1 + \alpha_{14}d_2 + \alpha_{33}d_3 + \alpha_{34}d_4 &= a_3 \\ \alpha_{14}d_1 - \alpha_{13}d_2 - \alpha_{34}d_3 + \alpha_{44}d_4 &= a_4 \end{aligned}\right\} \tag{A7}$$

where

$$\begin{aligned} \alpha_{11} &= \alpha_{22} = 1 - v - 4\left(1 - \frac{N_{Lc}}{A}\right) \\ \alpha_{33} &= \alpha_{44} = 2 - 4\left(\frac{1-v}{2} - \frac{N_{Lc}}{A}\right) \\ \alpha_{12} &= (1-v)q_1q_2 \\ \alpha_{13} &= (1+v)q_2 \\ \alpha_{14} &= (1+v)q_1 \\ \alpha_{34} &= 2q_1q_2. \end{aligned} \tag{A8}$$

Let a_1 and a_2 instead represent C_7 and C_8, and a_3 and a_4 represent C_3 and C_4, when eqn (37) is the right hand side of, respectively, the first and second eqns (32). Then for

$$a_1 \sinh q_1 Y \cos q_2 Y + a_2 \cosh q_1 Y \sin q_2 Y$$

at the right hand side of the first equation and

$$a_3 \sinh q_1 Y \sin q_2 Y + a_4 \cosh q_1 Y \cos q_2 Y$$

at the right hand side of the second equation of eqns (32), there are corresponding terms

$$d_1 \sinh q_1 Y \cos q_2 Y + d_2 \cosh q_1 Y \sin q_2 Y$$

in the solution $U_{2p}(Y)$ and terms

$$d_3 \sinh q_1 Y \sin q_2 Y + d_4 \cosh q_1 Y \cos q_2 Y$$

in the solution $V_{2p}(Y)$, respectively, where d_1–d_4 can be obtained by solving in the algebraic equations

$$\left.\begin{aligned} \alpha_{11}d_1 + \alpha_{12}d_2 - \alpha_{13}d_3 - \alpha_{14}d_4 &= a_1 \\ -\alpha_{12}d_1 + \alpha_{22}d_2 - \alpha_{14}d_3 + \alpha_{13}d_4 &= a_2 \\ -\alpha_{13}d_1 + \alpha_{14}d_2 + \alpha_{33}d_3 - \alpha_{34}d_4 &= a_3 \\ \alpha_{14}d_1 + \alpha_{13}d_2 + \alpha_{34}d_3 + \alpha_{44}d_4 &= a_4 \end{aligned}\right\} \tag{A9}$$

with the α_{ij} again defined by eqn (A8).

2. *Solution of the particular integrals of eqn* (39) *for* W_{2p}

Let a_1 represent any one of the four values C_1, C_2, C_3 and C_4 (or C_9, C_{10}, C_{11} and C_{12}), and b_1 represent the corresponding one of $\theta+q_1$, $\theta-q_1$, $\phi+q_1$ and $\phi-q_1$, when expression (46) is the right hand side of eqn (39). Then for each term at the right hand side of eqn (39) in the form of $a_1 \cosh b_1 Y$ (or $a_1 \sinh b_1 Y$), there is a corresponding term $d \cosh b_1 Y$ (or $d \sinh b_1 Y$) in the solution $W_{2p}(Y)$, where, for all four cases (i.e. $a_1 = C_1$, C_2, C_3 or C_4)

$$d = \frac{a_1}{b_1^4 - 8b_1^2 + 4\left(4 - \dfrac{N_{Lc}\lambda^2}{D\pi^2}\right)}. \tag{A10}$$

Let a_1 and a_2 instead represent C_5 and C_6, or C_7 and C_8, and p represent θ or ϕ, respectively. Then for

$$a_1 \sinh pY \sin q_2 Y + a_2 \cosh pY \cos q_2 Y$$

at the right hand side of eqn (39), there is a corresponding term in the solution of $W_{2p}(Y)$ equal to

$$d_1 \sinh pY \sin q_2 Y + d_2 \cosh py \cos q_2 Y \tag{A11}$$

where

$$d_1 = \frac{a_1 r + a_2 s}{r^2 + s^2}; \quad d_2 = \frac{a_2 r - a_1 s}{r^2 + s^2} \tag{A12}$$

and

$$r = p^4 + q_2^4 - 6p^2 q_2^2 - 8(p^2 - q_2^2) + 4\left(4 - \frac{N_{Lc}\lambda^2}{D\pi^2}\right)$$

$$s = 4(p^2 - q_2^2)pq_2 - 16pq_2. \tag{A13}$$

Let a_1 and a_2 instead represent C_{13} and C_{14}, or C_{15} and C_{16}, and p represent θ or ϕ, respectively. Then for

$$a_1 \sinh pY \cos q_2 Y + a_2 \cosh pY \sin q_2 Y$$

on the right hand side of eqn (39), there is a corresponding term in the solution of $W_{2p}(Y)$ equal to

$$d_1 \sinh pY \cos q_2 Y + d_2 \cosh pY \sin q_2 Y \tag{A14}$$

where

$$d_1 = \frac{a_1 r - a_2 s}{r^2 + s^2}; \quad d_2 = \frac{a_2 r + a_1 s}{r^2 + s^2} \tag{A15}$$

and r and s are again given by eqn (A13).